MTH 605: Topology I

Assignment 1

1 Problems for practice

1.1 Topological spaces and closed sets

(1) Show that the topologies $\mathbb{R}_\ell$ and $\mathbb{R}_K$ are not compatible.

(2) Describe a subbasis for the standard topology on $\mathbb{R}$ that is not a basis.

(3) Show that each of following collections define basis for a topology on X. Describe the topology generated in each case.

  (a) $\mathcal{B} = \{(a,b) \mid a < b,\ a \text{ and } b \text{ rational}\}$, $X = \mathbb{R}$.

  (b) $\mathcal{C} = \{[a,b) \mid a < b,\ a \text{ and } b \text{ rational}\}$, $X = \mathbb{R}$.

  (c) $\mathcal{D} = \{(a,b) \times (c,d) \mid a < b,\ c < d,\ a, b, c \text{ and } d \text{ rational}\}$, $X = \mathbb{R}^2$.

(4) If A, B, and A_α are subsets of a space X. Determine whether the following statements hold. Prove them if they are true, and give a counterexample if they are false.

  (a) If $A \subset B$, then $\overline{A} \subset \overline{B}$.

  (b) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

  (c) $\overline{\cup A_\alpha} \supset \cup \overline{A_\alpha}$.

  (d) $\overline{A \cap B} = \overline{A} \cap \overline{B}$.

  (e) $\overline{\cap A_\alpha} = \cap \overline{A_\alpha}$.

  (f) $\overline{A - B} = \overline{A} - \overline{B}$.

(5) If $A \subset X$, we define the boundary of A (denoted by ∂A) by $\partial A = \overline{A} \cap \overline{(X - A)}$. Show the following.

  (a) $A^\circ \cap \partial A = \emptyset$ and $\overline{A} = A^\circ \cup \partial A$.

(b) $\partial A = \emptyset$ if and only is A is both open and closed.
(c) U is open if and only if $\partial U = \overline{U} - U$.

(6) Find the ∂A and A°, if A is one of the following subsets of $\mathbb{R}^2$.

(a) $A = \mathbb{Q} \times \mathbb{R}$.
(b) $A = \{(x, y) \,|\, 0 < x^2 - y^2 \leq 1\}$.
(c) $A = \{(x, y) \,|\, x \neq 0 \text{ and } y = 1/x\}$.

1.2 Continuous functions, metric spaces, and product topology

(1) Show that for a function $f : \mathbb{R} \to \mathbb{R}$, the $\epsilon - \delta$ definition of continuity is equivalent to the open set definition.

(2) An indexed family of sets $\{A_\alpha\}$ is said to be *locally finite* if each point x of X has a neighborhood that intersects A_α for only finitely many values of α. Let $\{A_\alpha\}$ be a locally finite collection of closed subsets of X such that $X = \cup A_\alpha$. Show that if $f|_{A_\alpha}$ is continuous for each α, then f is continuous.

(3) If (X, d) is a metric space, then the topology induced by d is the coarsest topology relative to which the function d is continuous.

(4) Let $A \subset X$, and let $f : A \to Y$ be a continuous map of A into a Hausdorff space Y. Show that if f may be extended to a continuous function $g : \bar{A} \to Y$, then g is uniquely determined by f.

(5) Prove that an uncountable product of $\mathbb{R}$ with itself is not metrizable.

(6) Given $p \geq 1$, define

$$d(x, y) = \left[\sum_{i=1}^{n} |x_i - y_i|^p \right]^{1/p},$$

for $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{R}^n$. Show that d is a metric that induces the standard topology on $\mathbb{R}^n$.

(7) Let $\mathbb{R}_0$ be the subset of $\mathbb{R}^\infty$ consisting of sequences in $\mathbb{R}$ that are eventually 0. Find the closure of $\mathbb{R}_0$ in $\mathbb{R}^\infty$ under the product and box topologies.

(8) Define a map $h : \mathbb{R}^\infty \to \mathbb{R}^\infty$ that is linear in each coordinate. Is h continuous under the product and box topologies?

1.3 Quotient spaces, topological groups, and connectedness

(1) If $A \subset X$, a *retraction* of X onto A is a continuous map $r : X \to A$ such that $r(a) = a$ for each $a \in A$. Show that a retraction is a quotient map.

(2) Define an equivalence relation $\sim$ on $\mathbb{R}^2$ as follows: $(x_0, y_0) \sim (x_0, y_0)$ if $x_0 + {y_0}^2 = x_1 + {y_1}^2$. Describe the corresponding quotient space X^*.

(3) A *topological group* is a group $(G, \cdot)$ that is also a topological space satisying the T_1 axiom, such that the group operation $(g, h) \mapsto g \cdot h$ and the map $g \mapsto g^{-1}$ are both continuous maps. Show that $(\mathbb{R}, +)$, $\mathrm{GL}(n)$, and S^1 (seen as a subset of $\mathbb{C}$) are topological groups.

(4) Let G be a topological group, and let H be a subspace and a subgroup of G.

   (a) Show that both H and $\bar{H}$ are topological groups.
   (b) Give G/H the quotient topology using the lest cosets as partitions. Show that if H is closed in G, then the singletons are closed in G/H.
   (c) Show that $G \to G/H$ is open.
   (d) Show that if H is closed and $H \trianglelefteq G$, then G/H is a topological group.
   (e) Using (d), show that $\mathbb{R}/\mathbb{Z}$ is a topological group. Describe this space.

(5) If τ and τ' be two topologies on X such that $\tau \subset \tau'$. What does the connected of X in one topology imply in the other?

(6) A space is *totally disconnected* if its only connected subsets are the one-point sets. Show that if X has the discrete topology, then X is totally disconnected.

(7) Determine whether the following spaces are connected.

   (a) An infinite set with the cofinite topology.
   (b) $\mathbb{R}_\ell$.

(8) Let $p : X \to Y$ is a quotient map each of whose fibers are connected. Show that X is connected, whenever Y is connected.

(9) Using connectedness, establish the following facts.

(a) $(0,1)$, $(0,1]$, and $[0,1]$ are not homeomorphic.

(b) $\mathbb{R}^n$ and $\mathbb{R}$ are not homeomorphic for $n > 1$.

(10) Show that if $f : [0,1] \to [0,1]$ is a continuous map, then f has a fixed point.

(11) A space X is *weakly locally connected at* x if for every neighborhood U of x, there is a connected subspace of X contained in U that contains a neighborhood of x. Show that if X is weakly locally connected at every point, then X is locally connected.

(12) Describe the components and path components of the following spaces.

(a) $\mathbb{R}_\ell$

(b) $\mathbb{R}^\infty$ with product and box topologies

1.4 Compactness, Hausdorff spaces, and one-point compactification

(1) Show that X is Hausdorff if and only if the diagonal $\Delta = \{(x,x) \mid x \in X\}$ is closed in $X \times X$.

(2) Show that every compact subspace of a metric space is closed and bounded. Find a metric space in which the converse does not hold.

(3) Show that if X is compact Hausdorff under two topologies τ and τ', then either $\tau = \tau'$ or they are incomparable.

(4) Let Y be a compact space.

(a) Show that $\pi_1 : X \times Y \to X$ is a closed map.

(b) Let Y be a Hausdorff space, and let $f : X \to Y$. Then f is continuous if and only if the the *graph* of f, $G_f = \{(x, f(x)) \mid x \in X\}$ is closed in $X \times Y$.

(5) Show that a connected space having more than one point is uncountable.

(6) Let $p : X \to Y$ be a surjective continuous map each of whose fibers is compact. Show that if Y is compact, then X is compact.

(7) Let X be a compact Hausdorff space. Let $\mathcal{B}$ be a collection of closed connected subsets that are simply ordered under inclusion. Then show that $\cap_{a \in \mathcal{A}} A$ is connected.

(8) Establish the following facts.

   (a) $[0,1]$ is not compact in $\mathbb{R}_K$.
   (b) $\mathbb{R}_K$ is connected, but not path connected
   (c) $[0,1]$ is not limit point compact in $\mathbb{R}_\ell$.
   (d) Every subset of $\mathbb{R}$ under the cofinite topology is compact.
   (e) $\mathbb{Q}$ is not locally compact.

(9) A space X is *countably compact* if every countable covering of X has a finite subcovering. Show that in a T_1 space X, countable compactness is equivalent to limit point compactness. [Hint: If not finite subcollection of U_n covers X, then choose $x_n \notin U_1 \cup \ldots \cup U_n$ for each n.]

(10) Let (X, d) be a compact metric space. Show that every isometry on X is a homeomorphism.

(11) Let G be a topological group.

   (a) Show that if C is a component of G containing the identity element, then $C \trianglelefteq G$.
   (b) If G is locally compact and $H \leq G$, then G/H is locally compact.

(12) Show that a homeomorphism of locally compact Hausdorff spaces extends to their one-point compactification.

(13) Describe the one-point compactification of the following spaces.

   (a) $\mathbb{R}$
   (b) $\mathbb{Z}_+$
   (c) $\mathbb{R}^n$

(14) If $f, g : X \to Y$ be continuous maps and Y is Hausdorff, then show that the set $\{x \in X : f(x) = g(x)\}$ is closed in X.

2 Problems for submission

(Due 8/2/24)

- Solve problems 1.2 (6), 1.3 (10), 1.4 (4), and 1.4 (11) from the practice problems above.